Uncertainty Principles in Harmonic Analysis

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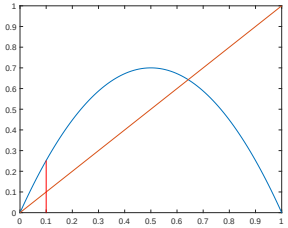

Overview

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   - Embedding into L^p spaces
   - Quaternionic Fourier transform
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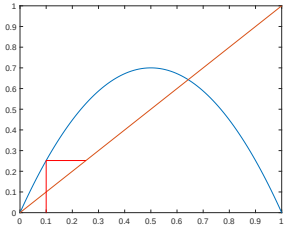


Quaternions

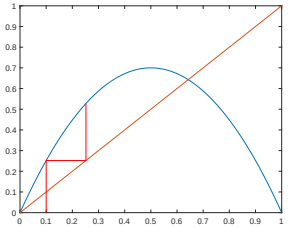
$$E := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I := \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix}, \quad J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K := \begin{bmatrix} 0 & \imath \\ \imath & 0 \end{bmatrix}$$

$$I^2 = J^2 = K^2 = -E \quad \text{and} \quad IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J.$$

Definition. The set

$$\mathbb{H} := \{aE + bI + cJ + dK \in \mathbb{R}^{2\times 2} : \ a, b, c, d \in \mathbb{R}\}$$

is referenced to as quaternions.



Schwartz space

We'll consider $f : \mathbb{R} \to \mathbb{H}$ functions, that is,

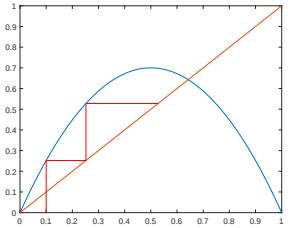
$$f(x) = f_1(x) + f_2(x)\mathbf{i} + f_3(x)\mathbf{j} + f_4(x)\mathbf{k} \qquad (x \in \mathbb{R}),$$

where

$$f_l : \mathbb{R} \to \mathbb{R} \qquad (l \in \{1, 2, 3, 4\}).$$

Definition.

$$\mathcal{S}(\mathbb{R}, \mathbb{H}) := \left\{ f \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{H}) : \|f\|_{k,l} := \sup_{x \in \mathbb{R}} \left| x^k f^{(\ell)}(x) \right| < \infty \quad (k, l \in \mathbb{N}_0) \right\}$$



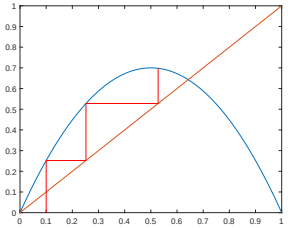
$L^p(\mathbb{R},\mathbb{H})$ space

Definition. We can also introduce the L^p norm for such functions via

$$\|f\|_p := \begin{cases} \|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}\mu_1(x)\right)^{1/p} & (p \in [1,+\infty)), \\ \|f\|_\infty := \operatorname{ess\,sup}(|f|) & (p=\infty) \end{cases}$$

And so we can consider the analogue of the L^p function space:

$$L^p(\mathbb{R},\mathbb{H}) := \left\{ f : \mathbb{R} \to \mathbb{H} : \|f\|_p < \infty \right\}.$$

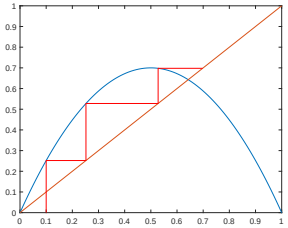


Relation between the spaces

We've shown that $\mathcal{S}(\mathbb{R},\mathbb{H})$ space can be embeddded into the L^p, i.e.

Theorem. For any $p \in [1,\infty]$ the following inclusion holds:

$$\mathcal{S}(\mathbb{R},\mathbb{H}) \subset L^p(\mathbb{R},\mathbb{H}).$$



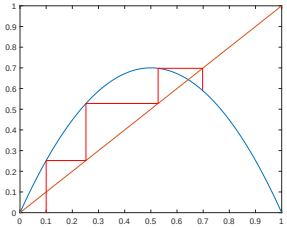
Quaternionic Fourier-transform

Definition. The one-dimensional quaternion Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$ is defined by the formula

$$(\mathcal{F}f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}} f(x) \exp(\mathbf{i}\xi x)\, \mathrm{d}\mu_1(x) \qquad (\xi \in \mathbb{R}).$$

Theorem (Fourier inversion). If $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$ then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{\mathbf{i}\xi x}\, \mathrm{d}\xi \qquad (x \in \mathbb{R}).$$



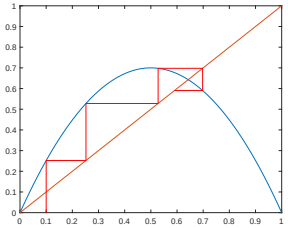
Two important properties

Theorem (Plancherel-like formula). If $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$ and $\widehat{f}$ denotes its' Fourier-transform, then

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} \left|\widehat{f}(\xi)\right|^2 \, d\xi.$$

Theorem. Let $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$, $n \in \mathbb{N}_0$ arbitrary number, and let $f^{(n)}$ denote the n-th derivative of f. Then the following holds:

$$\widehat{f^{(n)}}(\xi) = \widehat{f}(\xi) \cdot \xi^n \cdot (-\mathbf{i})^n.$$



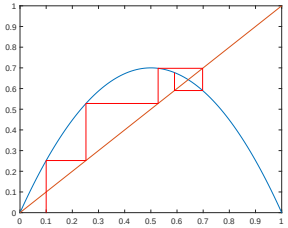
Donoho-Stark-like theorem

Theorem (Donoho-Stark-like). Let $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$, $\|f\| = 1$ and ε_1, ε_2 such nonnegative numbers that $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} < 1$. If $T, \Omega \subset \mathbb{R}$ sets are chosen such that

$$\int_T |f(x)|^2 \, dx \geq 1 - \varepsilon_1 \qquad \text{and} \qquad \int_T \left|\widehat{f}(\xi)\right|^2 d\xi \geq 1 - \varepsilon_2,$$

then the following holds:

$$\mu_1(T) \cdot \mu_1(\Omega) \geq \left(1 - \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}\right)^2.$$



Outline of Stein's method

- Define operators such that they are Hermitian adjoints, i.e.

$$\langle A_t f, g \rangle = \langle f, A_t^* g \rangle \qquad \forall\, f, g \in \mathcal{S}(\mathbb{R}, \mathbb{H}).$$

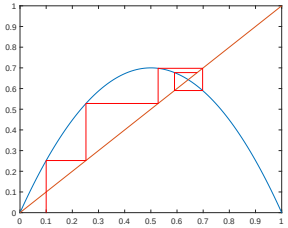
  and verify the following property holds:

$$\langle A_t^* A_t f, f \rangle \geq 0 \qquad \forall\, f \in \mathcal{S}(\mathbb{R}, \mathbb{H}).$$

- From the above inequality, obtain an upper bound for the integrals of f and f', e.g.

$$\left(\int_{\mathbb{R}} x^2 \left| f(x) \right|^2 \, \mathrm{d}x \right) \cdot \left(\int_{\mathbb{R}} \left| f'(x) \right|^2 \, \mathrm{d}x \right) \geq \frac{1}{4}$$

- Utilize the afromentioned theorems to switch between f' and $\widehat{f}$.



The result

Theorem. Suppose that $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$ and $\|f\|^2 = 1$. Then

$$\left(\int_{\mathbb{R}} x^2 \, |f(x)|^2 \, \mathrm{d}x \right) \cdot \left(\int_{\mathbb{R}} \left| \widehat{f}(\xi) \cdot \xi \cdot (-\mathbf{i}) \right|^2 \, \mathrm{d}\xi \right) \geq \frac{\pi}{2}.$$